Gravitational Wave in Media

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KMI film

https://www.youtube.com/watch?v=TP2WFsN_k6w

Introduction

Gravitational Waves

$\Leftarrow$ linearizing ($g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$) the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

by choosing the transverse gauge condition $\nabla^\mu h_{\mu\nu} = 0$,

$$\begin{aligned} 0 =& \frac{1}{2\kappa^2} \left(-\frac{1}{2} \left(-\Box^{(0)} h_{\mu\nu} - \nabla_\mu^{(0)} \nabla_\nu^{(0)} \left(g^{(0)\,\rho\lambda} h_{\rho\lambda} \right) \right. \right. \\ & \left. -2 R^{(0)\,\lambda\ \ \rho}_{\ \ \ \ \ \nu\ \ \mu} h_{\lambda\rho} + R^{(0)\,\rho}_{\ \ \ \ \ \mu} h_{\rho\nu} + R^{(0)\,\rho}_{\ \ \ \ \ \nu} h_{\rho\mu} \right) + \frac{1}{2} R^{(0)} h_{\mu\nu} \\ & \left. + \frac{1}{2} g_{\mu\nu}^{(0)} \left(-h_{\rho\sigma} R^{(0)\,\rho\sigma} - \Box^{(0)} \left(g^{(0)\,\rho\sigma} h_{\rho\sigma} \right) \right) \right) + \frac{1}{2} \delta T_{\mu\nu} \, , \end{aligned}$$

$\cdot T_{\mu\nu}$ depends on the metric. The dependence carries the informations on the mechanism of the expansion of the universe.

$\cdot \delta T_{\mu\nu}$ can be different in models even if the expansion history of the universe is identical.

S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov,
“Evolution of gravitons in accelerating cosmologies: The case of extended gravity”,
Phys. Rev. D **95** (2017) no.8, 083524
doi:10.1103/PhysRevD.95.083524
arXiv:1702.05517 [gr-qc]

S. Nojiri and S. D. Odintsov,
“Cosmological Bound from the Neutron Star Merger GW170817 in scalar-tensor and $F(R)$ gravity theories”,
Phys. Lett. B **779** (2018) 425
doi:10.1016/j.physletb.2018.01.078
arXiv:1711.00492 [astro-ph.CO].

K. Bamba, S. Nojiri and S. D. Odintsov,
“Propagation of gravitational waves in strong magnetic fields”,
Phys. Rev. D **98** (2018) no.2, 024002
doi:10.1103/PhysRevD.98.024002
arXiv:1804.02275 [gr-qc].

I. Brevik and S. Nojiri,
“Gravitational Waves in the Presence of Viscosity”,
arXiv:1901.00767 [gr-qc] (under submission).

Example: Scalar Tensor Theory

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi \,, \quad \mathcal{L}_\phi = -\frac{1}{2}\omega(\phi) g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \,,$$
$$\Rightarrow \; T_{\mu\nu} = -\omega(\phi)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi \,,$$
$$\Rightarrow \; \delta T_{\mu\nu} = \gamma_{\mu\nu}\mathcal{L}_\phi + \frac{1}{2}g_{\mu\nu}\omega(\phi)\partial^\rho\phi\partial^\lambda\phi\gamma_{\rho\lambda} \,,$$

Assuming a FRW spatially flat metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 ,$$

and $\phi = \phi(t)$, we may choose $\phi = t$
$\left(\phi = \phi\left(\tilde\phi\right) \Rightarrow \omega(\phi)\partial_\mu\phi\partial^\mu\phi = \tilde\omega\left(\tilde\phi\right)\partial_\mu\tilde\phi\partial^\mu\tilde\phi,\ \tilde\omega\left(\tilde\phi\right) \equiv \omega\left(\phi\left(\tilde\phi\right)\right)\phi'\left(\tilde\phi\right)^2\right)$,
the FRW equations $\left(H \equiv \frac{\dot a}{a}\right)$

$$\frac{3}{\kappa^2}H^2 = \frac{\omega}{2} + V , \quad -\frac{1}{\kappa^2}\left(2\dot H + 3H^2\right) = \frac{\omega}{2} - V ,$$
$$\Rightarrow \ \omega = -\frac{2}{\kappa^2}\dot H , \quad V = \frac{1}{\kappa^2}\left(\dot H + 3H^2\right) .$$

Then we can construct a model which reproduces any given history of expansion by $\omega\left(t \to \phi\right)$, $V\left(t \to \phi\right)$.

$$\text{ex. } a(t) = \left(\frac{t}{t_0}\right)^\alpha \Leftrightarrow \ \omega(\phi) = \frac{2\alpha}{\kappa^2 t_0^2 \phi^2} , \quad V(\phi) = \frac{3\alpha^2 - \alpha}{\kappa^2 t_0^2 \phi^2} .$$

t_0, α: real constants

$$\Leftrightarrow \quad \alpha = \frac{2}{3\left(1+w\right)}\,.$$

w: equation of state (EoS) parameter
(when Universe is filled with perfect fluid with $w = p/\rho$).

$w = 0 \Leftrightarrow$ dust $\sim$ cold dark matter (CDM)

$$\omega(\phi) = \frac{4}{3\kappa^2 t_0^2 \phi^2}\,, \quad V(\phi) = \frac{2}{3\kappa^2 t_0^2 \phi^2}\,.$$

$w = \frac{1}{3} \Leftrightarrow$ radiation

$$\omega(\phi) = \frac{1}{\kappa^2 t_0^2 \phi^2}\,, \quad V(\phi) = \frac{1}{4\kappa^2 t_0^2 \phi^2}\,.$$

$$
\begin{aligned}
\frac{\partial T_{\mu\nu}}{\partial g_{\rho\sigma}} =& \frac{1}{2}\left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho\right)\left(-\frac{1}{2} g^{\eta\zeta}\omega(\phi)\partial_\eta\phi\partial_\zeta\phi - V(\phi)\right) \\
&+ \frac{1}{2} g_{\mu\nu}\omega(\phi)\partial^\rho\phi\partial^\sigma\phi \,.
\end{aligned}
$$

In FRW universe, $\phi = \phi(t)$,

$$
\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{2}\left(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k\right)\left(\frac{1}{2}\dot\phi^2 - V(\phi)\right) \,.
$$

For dust $(w = 0)$,

$$
p = \frac{1}{2}\dot\phi^2 - V(\phi) = 0 \ \Rightarrow \ \frac{\partial T_{ij}}{\partial g_{kl}} = 0 \,,
$$

For radiation $(w = \frac{1}{3})$

$$
\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{2}\left(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k\right)\frac{1}{4\kappa^2 t_0^2 \phi^2} \,,
$$

Example: Quantum Thermodynamical Scalar Field

Free real scalar field ϕ with mass M in the flat background,

$$T_{00} = \rho = \frac{1}{2}\left(\pi^2 + \sum_{n=1,2,3}\left(\partial_n\phi\right)^2 + M^2\phi^2\right) ,$$

$$T_{ij} = \partial_i\phi\partial_j\phi + \frac{1}{2}\delta_{ij}\left(\pi^2 - \sum_{n=1,2,3}\left(\partial_n\phi\right)^2 - M^2\phi^2\right) .$$

$\pi = \dot{\phi}$: momentum conjugate to ϕ.

$$\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{4}\left(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k\right)\left(\pi^2 - \sum_{n=1,2,3}\left(\partial_n\phi\right)^2 - M^2\phi^2\right) + \frac{1}{2}\delta_{ij}\partial^k\phi\partial^l\phi .$$

At Finite Temperature T.

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_T = \frac{1}{12\pi^2}\delta_{ij}\delta^{kl}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+M^2}} \frac{\mathrm{e}^{-\beta\left(k^2+M^2\right)^{\frac{1}{2}}-i\mu}}{1-\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu}},$$

$$M \to 0 \ \Rightarrow \ \left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T,\,M=0} = \frac{1}{12\pi^2}\delta_{ij}\delta^{kl}\int_0^\infty dk \frac{k^3\mathrm{e}^{-\beta k-i\mu}}{1-\mathrm{e}^{-\beta k-i\mu}}.$$

Number density n:

$$\langle n\rangle_{T,\,M=0} = \frac{1}{2\pi^2}\int_0^\infty dk \frac{k^2\mathrm{e}^{-\beta\left(k^2+M^2\right)^{\frac{1}{2}}-i\mu}}{1-\mathrm{e}^{-\beta(k^2+M^2)^{\frac{1}{2}}-i\mu}}.$$

Dark matter: n might be fixed, $n=n_0$. $T\to 0$ (only ground state can contribute)

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T=0,n=n_0} = 0\,,$$

Tensor structures:

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_T \propto \delta_{ij}\delta^{kl} \ \Leftrightarrow \ \left. \frac{\partial T_{ij}}{\partial g_{kl}} \right|_{\text{Scalar Tensor Theory}} \propto \frac{1}{2}\left({\delta_i}^k {\delta_j}^l + {\delta_i}^l {\delta_j}^k \right)$$

$\Leftarrow$

$$\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{4}\left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k \right) \left(\pi^2 - \sum_{n=1,2,3} \left(\partial_n \phi\right)^2 - M^2 \phi^2 \right) + \frac{1}{2}\delta_{ij}\partial^k\phi\partial^l\phi \,.$$

In case of thermal quanta,

1st term$= 0$ by on-shell condition $\left(E^2 - k^2 - M^2 = 0\right)$,

2nd term $\sim \left\langle k^k k^l \right\rangle \propto \delta^{kl}$.

In case of scalar tensor theory $\left(M^2\phi^2 \Rightarrow V(\phi)\right)$,

$\phi = \phi(t) \Rightarrow$ 2nd term$= 0$.

When the number density of the particles is fixed $n = n_0$ and $T \to 0$ ($\Leftrightarrow$ Cold Dark Matter (CDM))

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T=0, n=n_0} = \left. \frac{\partial T_{ij}}{\partial g_{kl}} \right|_{\mathrm{Scalar-Tensor}} = 0 \,,$$

(c.f. R. Flauger and S. Weinberg, Phys. Rev. D **97** (2018) no.12, 123506) but in general,

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T} \neq \left. \frac{\partial T_{ij}}{\partial g_{kl}} \right|_{\mathrm{Scalar-Tensor}} \,,$$

for example, $w = \frac{1}{3}$ (radiation).

Propagation in GW170817

B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations],
"GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,"
Phys. Rev. Lett. **119** (2017) no.16, 161101
arXiv:1710.05832 [gr-qc]

Gravitational Wave from Neutron Star Merger

$$\left| \frac{c_{\mathrm{GW}}^2}{c^2} - 1 \right| < 6 \times 10^{-15} \,.$$

c: propagating speed of the light
c_{GW}: the propagating speed of the gravitational wave

$$0 = \nabla^\mu F_\mu{}^\nu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \right) = \nabla^2 A^\nu - \nabla^\nu \nabla^\mu A_\mu + R^{\mu\nu} A_\mu \,,$$

$$\Rightarrow$$

$$0 = \sum_{i=1,2,3} \partial_i \left(\partial_i A_t - \partial_t A_i \right) \,,$$

$$0 = \left(\partial_t + H \right) \left(\partial_i A_t - \partial_t A_i \right) + a^{-2} \left(\triangle A_i - \partial_i \sum_{j=1,2,3} \partial_j A_j \right) \,,$$

by assuming a spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i \right)^2 \,,$$

Landau gauge:
$0 = \nabla^\mu A_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} A_\nu \right) = -\partial_t A_t + 3HA_t + a^{-2} \sum_{i=1,2,3} \partial_i A_i$

$$\Rightarrow \ 0 = \nabla^2 A^\nu + R^{\mu\nu} A_\mu \,.$$

Assume $0 = A_t = \sum_{i=1,2,3} \partial_i A_i$

$$0 = - \left(\partial_t^2 + H \partial_t \right) A_i + a^{-2} \triangle A_i \,.$$

de Sitter space-time $H = h_0$, $a = \mathrm{e}^{h_0 t}$.
Assume $A_i \propto \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \times$ (the part only depending on t)
Replacing $\triangle$ by $-k^2 \equiv -\mathbf{k}\cdot\mathbf{k}$.
$s \equiv \mathrm{e}^{-h_0 t}$

$$\Rightarrow \quad 0 = \left(\frac{d^2}{ds^2} + \frac{k^2}{h_0^2}\right) A_i \,,$$

$$\Rightarrow \quad A_i = A_{i0} \cos\left(\frac{k}{h_0} s + \theta_0\right) \,.$$

Propagation in Scalar-Tensor Theory

Gravitational wave in de Sitter space-time realized by cosmological constant,

$$u = \frac{k}{h_0} s \,, \quad h_{ij} = s^{-\frac{1}{2}} l_{ij} \,.$$

$\Rightarrow$

$$0 = \left(\frac{d^2}{du^2} + \frac{1}{u} + 1 - \frac{\left(\frac{5}{2}\right)^2}{u^2} \right) l_{ij} \,,$$

Bessel's differential equation $\Rightarrow$ Bessel functions $J_{\pm\frac{5}{2}}(u)$.

Black hole/neutron star merger $s \equiv \mathrm{e}^{-h_0 t} \sim 1$. $\frac{k}{h_0} \gg 1$.

$$h_{ij} \sim \frac{1}{s} \cos \left(\frac{k}{h_0} s + \frac{\pm 5 + 1}{4} \pi \right) \,.$$

$\Rightarrow c = c_{\mathrm{GW}}$.

Power-law expansion $a(t) = \left(\frac{t}{t_0}\right)^\alpha$ in Scalar-Tensor Theory.

$$\omega(\phi) = \frac{2\alpha}{\kappa^2 t_0^2 \phi^2}\,, \quad V(\phi) = \frac{3\alpha^2 - \alpha}{\kappa^2 t_0^2 \phi^2}\,.$$

$\sim$ perfect fluid with a constant equation of state prameter w, $\alpha = \frac{2}{3(1+w)}$.

$$H = \frac{\alpha}{t}\,, \quad \dot{H} = -\frac{\alpha}{t^2}\,.$$

Black hole/neutron star merger $\Rightarrow$ $H \sim$ a constant, $H \sim h_0$.
$H^2 \sim \dot{H} \Rightarrow \dot{H} \sim$ a constant, $\dot{H} = h_1$

$$0 = \left(2\dot{H} + 6H^2 + H\partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) h_{ij}$$

$\Rightarrow$

$$0 = \left(\frac{d^2}{du^2} + \frac{1}{u} + 1 - \frac{\left(\frac{5}{2}\right)^2 - \frac{2h_1}{h_0^2}}{u^2} \right) l_{ij}\,,$$

Solution $J_{\pm\frac{5}{2}\sqrt{1-\frac{4h_1}{25h_0^2}}}(u)$.

$$h_{ij} \sim \frac{1}{s}\cos\left(\frac{k}{h_0}s + \frac{\pm 5\sqrt{1+\beta}+1}{4}\pi\right)\,, \quad \beta \equiv -\frac{4h_1}{25h_0^2}\,,$$

The propagation speed of the light is not changed.
The propagation speed of the gravitational wave is not changed, either.
The difference is in phase,

$$\beta = -\frac{4}{25\alpha} = -\frac{6(1+w)}{25}\,,$$

$$S_{F(R)} = \int d^4x\sqrt{-g}F(R)\,.$$

$F(R)$ gravity $\Leftrightarrow$ scalar-tensor theory, under the scale transformation (K. i. Maeda, Phys. Rev. D **39** (1989) 3159)

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{2\Phi} g_{\mu\nu}\,,$$

Because $\tilde{h}_i^{\ j} = \tilde{g}^{lj}\tilde{h}_{il} = \mathrm{e}^{-2\Phi} g^{lj} \mathrm{e}^{2\Phi} h_{il} = h_i^{\ j}$, $h_i^{\ j}$ results scale invariant in this sense.

Transverse and traceless gauge,

$$\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0$$

Scale transformation

$$\tilde{\nabla}^\mu \tilde{h}_\mu^{\ \nu} = \mathrm{e}^{-\Phi}\nabla^\mu h_{\mu\nu} + D\mathrm{e}^{-\Phi} g^{\mu\sigma} g^{\nu\rho}\Phi_{,\sigma} h_{\mu\rho} - \mathrm{e}^{-\Phi} g^{\nu\rho}\Phi_{,\rho} g^{\mu\sigma} h_{\mu\sigma}\,,$$
$$\Rightarrow\ \tilde{\nabla}^\mu \tilde{h}_\mu^{\ \nu} = D\mathrm{e}^{-\Phi} g^{\mu\sigma} g^{\nu\rho}\Phi_{,\sigma} h_{\mu\rho}\,.$$

D: the dimensions of space-time.

$$\widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = D\mathrm{e}^{-\Phi}\Phi_{,\mu} h^{\mu\nu}\,.$$

We assume that the background metric and therefore Φ only depend on the cosmological time t and also $g_{ti} = 0$.
$\Rightarrow$ when the perturbation with $h_{t\mu} = 0$,

$$\widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = \widetilde{g}^{\mu\nu}\widetilde{h}_{\mu\nu} = 0\,.$$

The gauge conditions for the graviton are not changed by the scale transformation.

Power law case,

$F(R) \sim R^m \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^\alpha \ \left(\alpha = -\frac{(m-1)(2m-1)}{m-2}\right)$

Scale transformation $F(R)$ gravity $\Rightarrow$ Scalar-Tensor Theory.

$$1 + \tilde{w}_{\mathrm{Scalar-Tensor}}$$

Speed of the propagation in the gravitational wave could not be changed by the scale transformation but there is a change of the phase

$$\beta = -\frac{4}{25\tilde{\alpha}} = \frac{12(m-2)^2}{25(m-1)^2} \sim \frac{243\,(1+w)^2}{25}\,,$$

(by assuming $w \sim -1$)

which is different from the case of the scalar-tensor theory.

$$\beta|_{\mathrm{Scalar-Tensor}} = -\frac{6(1+w)}{25}\,,$$

Gravitational Waves in a Viscous Fluid

I. Brevik and S. Nojiri,
"Gravitational Waves in the Presence of Viscosity," arXiv:1901.00767 [gr-qc]

Viscous fluid:
bulk viscosity ζ: related with the change of the volume
shear viscosity η: related with the shift of fluid

Bulk viscosity effectively generates negative pressure
$\rightarrow$ accelerating expansion of the universe
Shear viscosity makes the gravitational wave be enhanced or damp.

Gravitational wave in the present universe $\rightarrow$
the constraint on the shear viscosity

If the inflation in the early universe was generated by viscous fluid, the primordial gravitational wave might be enhanced or suppressed.

Cosmology with Viscous Fluid

S. Weinberg, "Entropy generation and the survival of protogalaxies in an expanding universe," Astrophys. J. **168** (1971) 175. doi:10.1086/151073

See also, for example,

I. Brevik, Ø. Grøn, J. de Haro, S. D. Odintsov and E. N. Saridakis, "Viscous Cosmology for Early- and Late-Time Universe," Int. J. Mod. Phys. D **26** (2017) no.14, 1730024 doi:10.1142/S0218271817300245 [arXiv:1706.02543 [gr-qc]].

K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, "Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests," Astrophys. Space Sci. **342** (2012) 155 doi:10.1007/s10509-012-1181-8 [arXiv:1205.3421 [gr-qc]].

⋮

Characteristics of a Viscous Fluid

Viscous fluid with four velocity (U^μ)
($U^0 = 1, U^i = 0$ in the comoving frame)

Projection tensor $\Rightarrow$ spatial directions perpendicular to U^μ

$$\gamma_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \,.$$

Rotation tensor $\omega_{\mu\nu}$, Expansion tensor $\theta_{\mu\nu}$:

$$\omega_{\mu\nu} \equiv \frac{1}{2}\left(U_{\mu;\alpha}\gamma^\alpha_\nu - U_{\nu;\alpha}\gamma^\alpha_\mu \right) \,,$$
$$\theta_{\mu\nu} \equiv \frac{1}{2}\left(U_{\mu;\alpha}\gamma^\alpha_\nu + U_{\nu;\alpha}\gamma^\alpha_\mu \right) \,,$$

Scalar expansion θ: $\theta \equiv \theta^\mu_\mu = U^\mu{}_{;\mu}$.

In the FRW space-time,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \quad \Rightarrow \quad \theta = 3H\,.$$

Shear tensor

$$\sigma_{\mu\nu} \equiv \theta_{\mu\nu} - \frac{\theta}{3}\gamma_{\mu\nu} = \theta_{\mu\nu} - H\gamma_{\mu\nu}\,. \quad \left(\sigma^\mu_\mu = 0\right)$$

Assuming temperature T is constant,
Energy-momentum tensor $T_{\mu\nu}$ of fluid

$$T_{\mu\nu} = \rho U_\mu U_\nu + \left(p - \zeta\theta\right)\gamma_{\mu\nu} - 2\eta\sigma_{\mu\nu}\,.$$

Effective pressure

$$p_{\mathrm{eff}} \equiv p - \zeta\theta = p - 3\zeta H\,,$$

Lower than p because $\zeta \geq 0 \Leftarrow$ a consequence of thermodynamics.

Cosmology in the Presence of a Viscous Fluid

FRW equations

$$\frac{3}{\kappa^2} H^2 = \rho \,, \quad -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right) = p - 3\zeta H \,,$$

$\Rightarrow$ conservation law,

$$0 = \dot{\rho} + 3H\left(\rho + p - 3\zeta H\right) \,.$$

Shear viscosity does not contribute to the background evolution.

Assume $p \propto \rho$ and $\zeta \propto \rho^{\lambda}$ with a constant λ

$$p = w\rho \,, \quad \zeta = \tilde{\zeta}_0 \rho^{\lambda} \,.$$

w: equation of state (EoS) parameter, $\tilde{\zeta}_0$: constant

$$\Rightarrow \quad 0 = \dot{\rho} + \kappa\left(w+1\right) 3^{\frac{1}{2}} \rho^{\frac{3}{2}} - 3\kappa^2 \tilde{\zeta}_0 \rho^{\lambda+1} \,,$$

Especially in case of $\lambda = \frac{1}{2}$, which is often chosen,

$$\begin{aligned} w_{\mathrm{eff}} =& w - 3^{\frac{1}{2}}\kappa\tilde{\zeta}_0 \,, \\ \Rightarrow \quad 0 =& \dot\rho + \kappa \left(w_{\mathrm{eff}} + 1\right) 3^{\frac{1}{2}} \rho^{\frac{3}{2}} = \dot\rho + 3H \left(w_{\mathrm{eff}} + 1\right) \rho \,, \end{aligned}$$

Then even if $w > 0$, in case $w_{\mathrm{eff}} < -\frac{1}{3}$, the accelerating expansion of the universe is generated.

Due to the expansion, internal energy of fluid decreases by positive pressure but increases by the bulk viscosity.

In the FRW space-time, (t,t), (i,j), (t,i) components

$$
\begin{aligned}
0 =& \frac{1}{2\kappa^2}\left(\frac{1}{2}\Box^{(0)} h_{tt} + \frac{1}{2}\partial_t^2\left(g^{(0)\,\rho\lambda} h_{\rho\lambda}\right) + \frac{1}{2}\Box^{(0)}\left(g^{(0)\,\rho\sigma} h_{\rho\sigma}\right)\right. \\
& \left. -\frac{1}{2}\left(\dot H - H^2\right)\left(g^{(0)\,ij} h_{ij}\right) - \frac{3}{2}\left(\dot H - H^2\right) h_{tt}\right) + \frac{1}{2}\delta T_{\mathrm{matter}\,tt}\,, \\
0 =& \frac{1}{2\kappa^2}\left(\frac{1}{2}\Box^{(0)} h_{ij} + \frac{1}{2}\left(\partial_i\partial_j - H\delta_{ij}\partial_t\right)\left(g^{(0)\,\rho\lambda} h_{\rho\lambda}\right)\right. \\
& - \frac{1}{2} g^{(0)}_{ij}\Box^{(0)}\left(g^{(0)\,\rho\sigma} h_{\rho\sigma}\right) + \frac{1}{2}\left(\dot H + H^2\right) g^{(0)}_{ij} h_{tt} \\
& \left. +2\left(\dot H + H^2\right) h_{ij} - \frac{1}{2} g^{(0)}_{ij}\left(\dot H + H^2\right)\left(g^{(0)\,kl} h_{kl}\right)\right) + \frac{1}{2}\delta T_{\mathrm{matter}\,ij}\,, \\
0 =& \frac{1}{2\kappa^2}\left(\frac{1}{2}\Box^{(0)} h_{ti} + \frac{1}{2}\nabla^{(0)}_t\nabla^{(0)}_i\left(g^{(0)\,\rho\lambda} h_{\rho\lambda}\right) + \left(2\dot H + 4H^2\right) h_{ti}\right) \\
& + \frac{1}{2}\delta T_{\mathrm{matter}\,ti}\,.
\end{aligned}
$$

Energy-momentum tensor $\delta T_{\mu\nu}$ for viscous fluid
The metric dependence of energy density ρ and pressure p

$$\delta\rho = \rho^{\mu\nu} h_{\mu\nu} \,, \quad \delta p = p^{\mu\nu} h_{\mu\nu} \,.$$

Shear viscosity η and bulk viscosity ζ may also depend on the energy density ρ and the pressure p.

$$\begin{aligned}\delta\eta =& \eta^{(\rho)}\delta\rho + \eta^{(p)}\delta p = \eta^{\mu\nu} h_{\mu\nu} \equiv \left(\eta^{(\rho)}\rho^{\mu\nu} + \eta^{(p)} p^{\mu\nu}\right) h_{\mu\nu} \,, \\ \delta\zeta =& \zeta^{(\rho)}\delta\rho + \zeta^{(p)}\delta p = \zeta^{\mu\nu} h_{\mu\nu} \equiv \left(\zeta^{(\rho)}\rho^{\mu\nu} + \zeta^{(p)} p^{\mu\nu}\right) h_{\mu\nu} \,.\end{aligned}$$

In the FRW space-time, we may assume,
$\rho^{\mu\nu}$, $p^{\mu\nu}$, $\eta^{\mu\nu}$, and $\zeta^{\mu\nu}$ only depend on t and not on $\left(x^i\right)$.

$$\begin{aligned}U^\mu U_\mu = -1 \ &\Rightarrow 0 = 2\left(\delta U^\mu\right) + U^\mu U^\nu h_{\mu\nu} = U^\mu \left(2 g_{\mu\nu}\delta U^\nu + h_{\mu\nu} U^\nu\right) \,, \\ &\Rightarrow \delta U^\mu = -\frac{1}{2} g^{\mu\rho}\left(h_{\rho\nu} U^\nu + l_\rho\right) \,, \quad l_\rho : \ U^\mu l_\mu = 0 \,.\end{aligned}$$

$\Rightarrow$

$$
\begin{aligned}
\delta\theta_{\mu\nu} =& \frac{1}{2}\left(\delta U_{\mu;\alpha}\gamma^{\alpha}_{\ \nu} - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\mu h_{\alpha\lambda} + \nabla_\alpha h_{\mu\lambda} - \nabla_\lambda h_{\mu\alpha}\right)U_\kappa\gamma^{\alpha}_{\ \nu}\right. \\
& - U_{\mu;\alpha}U^\alpha h_{\nu\xi}U^\xi - U_{\mu;\alpha}U^\alpha g_{\nu\xi}\delta U^\xi - U_{\mu;\alpha}\delta U^\alpha U_\nu \\
& + \delta U_{\nu;\alpha}\gamma^{\alpha}_{\ \mu} - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\nu h_{\alpha\lambda} + \nabla_\alpha h_{\nu\lambda} - \nabla_\lambda h_{\nu\alpha}\right)U_\kappa\gamma^{\alpha}_{\ \mu} \\
& \left. - U_{\nu;\alpha}U^\alpha h_{\mu\xi}U^\xi - U_{\nu;\alpha}U^\alpha g_{\mu\xi}\delta U^\xi - U_{\nu;\alpha}\delta U^\alpha U_\mu\right) , \\
\delta\theta =& - g^{\rho\mu}h_{\mu\nu}g^{\nu\sigma}\theta_{\rho\sigma} + \delta U_{\mu;\alpha}\gamma^{\mu\alpha} - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\mu h_{\alpha\lambda} + \nabla_\alpha h_{\mu\lambda} - \nabla_\lambda h_{\mu\alpha}\right)U_\kappa\gamma^{\alpha\mu} \\
& - g^{\mu\nu}U_{\mu;\alpha}U^\alpha h_{\nu\xi}U^\xi - g^{\mu\nu}U_{\mu;\alpha}U^\alpha g_{\nu\xi}\delta U^\xi - U_{\mu;\alpha}\delta U^\alpha U^\mu ,
\end{aligned}
$$

$$
\begin{aligned}
\delta\sigma_{\mu\nu} =& \delta\theta_{\mu\nu} - \frac{1}{3}\delta\theta\gamma_{\mu\nu} - \frac{\theta}{3}\left(h_{\mu\nu} + \delta U_\mu U_\nu + U_\mu \delta U_\nu\right) \\
=& \frac{1}{2}\left(\delta U_{\mu;\alpha}\gamma^\alpha_{\ \nu} - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\mu h_{\alpha\lambda} + \nabla_\alpha h_{\mu\lambda} - \nabla_\lambda h_{\mu\alpha}\right) U_\kappa \gamma^\alpha_{\ \nu} \right. \\
& - U_{\mu;\alpha} U^\alpha h_{\nu\xi} U^\xi - U_{\mu;\alpha} U^\alpha g_{\nu\xi}\delta U^\xi - U_{\mu;\alpha}\delta U^\alpha U_\nu \\
& + \delta U_{\nu;\alpha}\gamma^\alpha_{\ \mu} - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\nu h_{\alpha\lambda} + \nabla_\alpha h_{\nu\lambda} - \nabla_\lambda h_{\nu\alpha}\right) U_\kappa \gamma^\alpha_{\ \mu} \\
& \left. - U_{\nu;\alpha} U^\alpha h_{\mu\xi} U^\xi - U_{\nu;\alpha} U^\alpha g_{\mu\xi}\delta U^\xi - U_{\nu;\alpha}\delta U^\alpha U_\mu \right) \\
& - \frac{1}{3}\left(- g^{\rho\eta} h_{\eta\zeta} g^{\zeta\sigma}\theta_{\rho\sigma} + \delta U_{\eta;\alpha}\gamma^{\eta\alpha} \right. \\
& - \frac{1}{2}g^{\kappa\lambda}\left(\nabla_\eta h_{\alpha\lambda} + \nabla_\alpha h_{\eta\lambda} - \nabla_\lambda h_{\eta\alpha}\right) U_\kappa \gamma^{\alpha\eta} \\
& \left. - g^{\eta\zeta} U_{\eta;\alpha} U^\alpha h_{\zeta\xi} U^\xi - g^{\eta\zeta} U_{\eta;\alpha} U^\alpha g_{\zeta\xi}\delta U^\xi - U_{\eta;\alpha}\delta U^\alpha U^\eta \right)\gamma_{\mu\nu} \\
& - \frac{\theta}{3}\left(h_{\mu\nu} + \delta U_\mu U_\nu + U_\mu \delta U_\nu\right) .
\end{aligned}
$$

$$\begin{aligned}\delta T_{\mu\nu} =& \delta\rho U_\mu U_\nu + (\rho + p - \zeta\theta)\, \delta U_\mu U_\nu + (\rho + p - \zeta\theta)\, U_\mu \delta U_\nu \\ & + (\delta p - \delta\zeta\theta - \zeta\delta\theta)\, \gamma_{\mu\nu} \\ & + (p - \zeta\theta)\, h_{\mu\nu} - 2\delta\eta\sigma_{\mu\nu} - 2\eta\delta\sigma_{\mu\nu} \,.\end{aligned}$$

Case of Bulk Viscosity

First ignore shear viscosity $\eta = 0$.
massless spin two mode

$$h_{it} = h_{ti} = 0\,, \quad \sum_{i=1,2,3} h_{ii} = 0\,, \quad i = 1,2,3\,, \quad h_{tt} = 0$$

$\Rightarrow \delta U^\mu = 0$ ($\delta U^\mu = -\frac{1}{2} g^{\mu\rho} \left(h_{\rho\nu} U^\nu + l_\rho \right)$, assume $l_\mu = 0$).

If we assume ρ^{ij}, $p^{ij} \propto \delta^{ij}$, $\delta\rho = \delta p = \delta\zeta = \delta\eta = 0$
$(\delta\rho = \rho^{\mu\nu} h_{\mu\nu}\,, \quad \delta p = p^{\mu\nu} h_{\mu\nu})$.

$$\theta_{tt} = \theta_{ti} = \theta_{it} = U_{tt} = U_{t;i} = U_{i;t} = 0\,, \quad \theta_{ij} = U_{i;j} = a^2 H \delta_{ij}\,, \quad \theta = 3H\,.$$

$$\Rightarrow \delta\theta_{tt} = \delta\theta_{ti} = \delta\theta_{it} = 0\,, \quad \delta\theta_{ij} = \frac{1}{2}\left(2\nabla_i h_{jt} - \nabla_t h_{ij}\right) = -\frac{1}{2}\partial_t h_{ij}\,,$$

$$\delta\theta = 0\,,$$

$$\Rightarrow \delta T_{tt} = \delta T_{ti} = \delta T_{it} = 0\,, \quad \delta T_{ij} = \left(p - 3H\zeta\right) h_{ij}\,.$$

(t,t) and (t,i) components of equation for gravitational wave are trivially satisfied.

(i,j) component

$$
\begin{aligned}
0 =& \frac{1}{2\kappa^2}\left(\frac{1}{2}\Box^{(0)} h_{ij} + 2\left(\dot{H} + H^2\right) h_{ij}\right) + \frac{1}{2}\left(p - 3H\zeta\right) h_{ij} \\
=& \frac{1}{2\kappa^2}\left(\frac{1}{2}\left(-\partial_t^2 h_{ij} + a^{-2}\triangle h_{ij}\right) + \left(3\dot{H} + 4H^2\right) h_{ij}\right) \\
& + \frac{1}{2}\left(p - 3H\zeta\right) h_{ij} \,,
\end{aligned}
$$

Massless spin two mode (gravitational wave) exists even if $\zeta \neq 0$.

c.f.

G. Goswami, G. K. Chakravarty, S. Mohanty and A. R. Prasanna, "Constraints on cosmological viscosity and self interacting dark matter from gravitational wave observations," Phys. Rev. D **95** (2017) no.10, 103509 doi:10.1103/PhysRevD.95.103509 [arXiv:1603.02635 [hep-ph]].

Case of Shear Viscosity

$\eta \neq 0$.
Under assumption

$$h_{it} = h_{ti} = 0\,, \quad \sum_{i=1,2,3} h_{ii} = 0\,, \quad i = 1,2,3\,,$$
$$\Rightarrow \delta U^{\mu} = 0\,.$$

$$\Rightarrow \ \delta\sigma_{tt} = \delta\sigma_{ti} = 0\,,$$
$$\delta\sigma_{ij} = -\frac{1}{2}\partial_t h_{ij} - H h_{ij}\,.$$

(t,t), (t,i) components of gravitational wave equation are trivially satisfied, again.

(i,j) component

$$0 = \frac{1}{2\kappa^2}\left(\frac{1}{2}\left(-\partial_t^2 h_{ij} + a^{-2}\triangle h_{ij}\right) + \left(3\dot{H} + 4H^2\right) h_{ij}\right) + \frac{1}{2}\left(p - 3H\zeta\right) h_{ij} - 2\eta\left(-\frac{1}{2}\partial_t h_{ij} - H h_{ij}\right) ,$$

Enhancement ($\eta > 0$, usual) or dissipation ($\eta < 0$) of the gravitational wave.

Cosmological speculations

Present Universe

Recently observed gravitational waves

B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116** (2016) 061102 doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence," Phys. Rev. Lett. **116** (2016) 241103 doi:10.1103/PhysRevLett.116.241103 [arXiv:1606.04855 [gr-qc]].

B. P. Abbott *et al.* [LIGO Scientific and VIRGO Collaborations], "GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2," Phys. Rev. Lett. **118** (2017) 221101 doi:10.1103/PhysRevLett.118.221101 [arXiv:1706.01812 [gr-qc]].

⋮

Distances $\sim$ a few hundreds Mpc.

No dissipation or enhancement has not been observed
$\Rightarrow$ constraint on the shear viscosity η_0 in the present universe

$$\left|\kappa^2\eta_0\right| \ll \left(10^3\,\mathrm{Mpc}\right)^{-1}\ ,$$

1 Mpc$=3.086\times 10^{22}$ m, 1 s$=3\times 10^8$ m, $\kappa^2 = 1.87\times 10^{-26}$ m/kg,

$$|\eta_0| \ll 5\times 10^8\ \mathrm{Pa\ s.}$$

If the dark energy is represented by the viscous fluid, bulk viscosity ζ

$$\frac{1}{\kappa^2\zeta_0} \sim \frac{1}{H} \sim 10^4\,\mathrm{Mpc}\,.$$

or

$$\zeta_0 \sim 5\times 10^7\ \mathrm{Pa\ s}\,.$$

Inflation

If inflation in the early universe could be generated by viscous fluid, we may expect that the shear viscosity could be large.

Period of the inflation could be estimated to be 10^{-34} sec.
$\sim 10^{19}\,\mathrm{eV} = 10^{10}\,\mathrm{GeV}$.
If primordial gravitational wave is detected,

$$\left|\eta\kappa^2\right| \ll 10^{10}\,\mathrm{GeV}\,.$$

If the scale of the inflation is the GUT scale $\sim 10^{15}\,,\mathrm{GeV}$,

$$\left|\kappa^2\zeta\right| \sim H \sim 10^{11}\,\mathrm{GeV}\,.$$

If $|\zeta| \sim |\eta|$, primordial gravitational wave may be enhanced or absorbed into the viscous fluid.

How about QCD plasma?

Summary

- We discussed the evolution of cosmological gravitational wave showing how the cosmological background affects their dynamics.
- The detection of cosmological gravitational wave could constitute an extremely important signature to discriminate among different cosmological models.
- We considered the cases of scalar-tensor gravity and $F(R)$ gravity where it is demonstrated the amplification of graviton amplitude changes if compared with General Relativity.
- We also show the speed of the gravitational wave by the modified gravity does not change different from the scalar tensor theory without higher derivative couplings.
- We derived the governing equation for gravitational waves propagating in a fluid with bulk viscosity ζ and shear viscosity η.
  - Bulk viscosity ζ: Not so strong affect on the propagation
  - Shear viscosity η: enhancement or dissipation of the gravitational wave
    $\Rightarrow$ Primordial gravitational wave might be enhanced or suppressed.

- K. Bamba, S. Nojiri and S. D. Odintsov,
  "Propagation of gravitational waves in strong magnetic fields,"
  Phys. Rev. D **98** (2018) no.2, 024002
  doi:10.1103/PhysRevD.98.024002
  arXiv:1804.02275 [gr-qc].
  Enhancement of gravitational wave by strong magnetic fields
  - NGC 6946 size $\sim 100\,\mathrm{k}$ light years, $B \sim \mu\mathrm{G}$
  - Magnestar (e.g. SGR 1806-20) very strong magnetic field $10^{11}\,\mathrm{T}$.

謝謝!